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A PROBLEM OF OPTIMAL PURSUIT IN SYSTEMS WITH DISTRIBUTED PARAMETERS†

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A game-theoretical problem whose dynamics is described by a partial differential equation is considered. The players' controls, representing additively the right-hand side of the equation, are subject to integral constraints. The goal of the pursing player, who possesses information on the instantaneous value of the evader's control, is to bring the system to an undisturbed state. To solve the problem, the method of system decomposition developed in [1] for a controlled system is used. The optimal pursuit time is found and the players' optimal controls are constructed in explicit form. © 2003 Elsevier Science Ltd. All rights reserved.

In an earlier paper [2], a condition was established for the pursuit problem considered here to be solvable. This paper is as continuation of [1–5].

1. FORMULATION OF THE PROBLEM

Let A be a differential operator defined in the space $L_2(\Omega)$ as follows [1]:

$$Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial z}{\partial x_j}\right), \; x \in \Omega; \; a_{ij}(x) = a_{ji}(x) \in C^1(\overline{\Omega}) \tag{1.1}$$

where Ω is a bounded domain in R^n. The domain of definition $D(A)$ of the operator A is $\overset{\circ}{C}{}^2(\Omega)$ (the space of twice continuously differentiable functions with compact support). The coefficients a_{ij} satisfy the following condition: A constant $\gamma \neq 0$ exists such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant \gamma^2 \sum_{i=1}^{n} \xi_i^2, \; (\xi_1,\dots,\xi_n) \in R^n, \; x \in \Omega \tag{1.2}$$

which means that A is an elliptic operator. Putting

$$(z, y)_A = (Az, y), \; z, y \in \overset{\circ}{C}{}^2(\Omega)$$

it can be shown that $(.,.)_A$ satisfies all the requirements for a scalar product. Thus, $\overset{\circ}{C}{}^2(\Omega)$ becomes a Hilbert space, but it is not complete relative to the norm generated by the scalar product $(.,.)_A$. Completing the space $\overset{\circ}{C}{}^2(\Omega)$ relative to the norm

$$\| z \|_A = \sqrt{(Az, z)}, \; z \in \overset{\circ}{C}{}^2(\Omega)$$

we obtain a complete Hilbert space, called the energy space of the operator A and denoted by H_A.

It is well known [6] that an operator A satisfying condition (1.2) has a discrete spectrum, that is, an infinite sequence $\lambda_1, \lambda_2, \dots, 0 < \lambda_1 \leqslant \lambda_2 \leqslant \dots$ of generalized eigenvalues exists with a unique limit at infinity, and also a sequence $\varphi_1, \varphi_2, \dots$ of generalized eigen elements, which is complete in the space L2(Ω). Without loss of generality, we may assume that $(\varphi_i, \varphi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta.

Using these data, we construct the following spaces (which are of course associated with the operator A) [7]. Let

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$$l_r = \left\{ \alpha = (\alpha_1, \alpha_2, \ldots) : \sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 < \infty \right\}, \quad H_r(\Omega) = \left\{ f \in L_2(\Omega) : f = \sum_{i=1}^{\infty} a_i \varphi_i, \ \alpha \in l_r \right\}$$

We define products in the spaces l_r and $H_r(\Omega)$ as follows:

$$(\alpha, \beta)_r = \sum_{i=1}^{\infty} \lambda_i^r \alpha_i \beta_i, \alpha, \ \beta \in l_r \quad \left(\| \alpha \| = \left(\sum_{i=1}^{\infty} \lambda_i^r \alpha_i^2 \right)^{1/2} \right)$$

Similarly

$$(f, g)_r = (\alpha, \beta)_r; \ f = \sum_{i=1}^{\infty} \alpha_i \varphi_i, \ g = \sum_{i=1}^{\infty} \beta_i \varphi_i \ (\| f \| = \| \alpha \|)$$

Note that $H_0(\Omega) = L_2(\Omega)$ and $H_r(\Omega) \subset H_s(\Omega)$ for $s < r$.

Let $C(0, T; H_r(\Omega))$ ($L_2(0, T; H_r(\Omega))$) denote the space of all continuous (square summable) functions in $[0, T]$ with values in $H_r(\Omega)$, where T is a positive constant.

The equations arising in what follows will be understood in the sense of the theory of distributions (generalized functions) [7].

Consider the differential game

$$\begin{gathered} dz(t)/dt + Az(t) = -u(t) + v(t) \\ u(\cdot), v(\cdot) \in L_2(0, T; \ H_r(\Omega)), \ z(0) = z_0 \neq 0, \ z_0 \in H_{r+1}(\Omega) \end{gathered} \tag{1.3}$$

The operator A is given in the form of (1.1).

It was proved in [7] that problem (1.3) has a unique solution $z(t)$ in the space $C(0, T; H_{r+1}(\Omega))$, if $z_0 \in H_{r+1}(\Omega)$ for some $r \geqslant 0$.

The function $u(t)$, $v(t)$, $0 < t \leqslant T$, will be called the controls of the first player (the pursuer) and the second player (the evader), respectively. They are subject to the constraints

$$\|u(\cdot)\| \leqslant \rho, \ \|v(\cdot)\| \leqslant \sigma \tag{1.4}$$

where ρ and σ are non-negative constants. Controls $u(t)$, $v(t)$, $0 \leqslant t \leqslant T$ satisfying conditions (1.4) are said to be admissible.

Definitions. 1. We shall say that a pursuit can be completed in game (1.3), (1.4) from initial point z_0 in time $T = T(z_0) \geqslant 0$ if one can choose a value $u(t)$ in such a way that, for any admissible control $v(\cdot)$ of the evader, $z(t') = 0$ for some $t' \in [0, T]$, where $z(\cdot)$ is a solution of Eq. (1.3) for the controls u and v. Under these conditions, to construct the pursuer's control $u(t)$ at each instant of time $t \in [0, T]$, it is permissible to use $z(t)$ and the quantity $v(t)$. The number T is called the guaranteed pursuit time. The pursuer's goal is to minimize the guaranteed pursuit time, while the evader's goal is to maximize it.

2. If an admissible control of the evader $v(\cdot)$ exists such that, for any admissible control $u(\cdot)$ of the pursuer, $z(t) \neq 0$ for $t \in [0, T]$, then the guaranteed pursuit time T is called the optimal pursuit time. Under those conditions, to construct $v(t)$ at every instant of time $t \in [0, T]$ it is permissible to use $z(t)$ and the quantities

$$\rho(t) = (\rho^2 - F(t))^{1/2}, \ \sigma(t) = (\sigma^2 - G(t))^{1/2}$$

where

$$F(t) = \sum_{i=1}^{\infty} \lambda_i^r \int_0^t u_i^2(s) ds, \ G(t) = \sum_{i=1}^{\infty} \lambda_i^r \int_0^t v_i^2(s) ds$$

$u_i(t) = (u(t), \varphi_i)$, $v_i(t) = (v(t), \varphi_i)$ $(i = 1, 2, \ldots)$ are the Fourier coefficients of the functions $u(t)$ and $v(t)$.

Problem. For every initial point, it is required to find the optimal pursuit time in the game (1.3), (1.4).

2. SOLUTION OF THE PROBLEM

Let $t = \vartheta$ be a solution of the equation

$$\Sigma(t) \doteq \lambda_1^r r_1(t) z_{10}^2 + \lambda_2^r r_2(t) z_{20}^2 + \ldots = (\rho - \sigma)^2, \quad r_i(t) = 2\lambda_i / (e^{2\lambda_i t} - 1) \tag{2.1}$$

where $z_{k0} = (z_0, \varphi_k)$ are the Fourier coefficients of the function z_0.

It can be verified that the left-hand side of Eq. (2.1) is a decreasing function of the variable t, $t \geqslant 0$, which tends to zero as $t \to \infty$ and to infinity as $t \to 0$. Consequently, Eq. (2.1) has a unique solution.

Theorem. If $\rho > \sigma$, then the number ϑ is the optimal pursuit time in the game (1.3), (1.4).

Proof. 1. *Construction of the pursuer's control.* Let $v(\cdot)$ be an arbitrary control of the evader. Let us express $z(t)$ and the functions $u(t)$ and $v(t)$ as Fourier series

$$z(t) = \sum_{k=1}^{\infty} z_k(t)\varphi_k, \quad \sum_{k=1}^{\infty} \lambda_k^{r+1} \int_0^{\vartheta} |z_k(t)|^2 \, dt < \infty, \quad z_k(\cdot) \in L_2 \tag{2.2}$$

$$u(t) = \sum_{k=1}^{\infty} u_k(t)\varphi_k, \quad F(\vartheta) \leqslant \rho^2, \quad v(t) = \sum_{k=1}^{\infty} v_k(t)\varphi_k, \quad G(\vartheta) \leqslant \sigma^2, \quad u_k(\cdot), \ v_k(\cdot) \in L_2 \tag{2.3}$$

Substituting expansions (2.2) and (2.3) into Eq. (1.3) and equating the corresponding coefficients of the complete system $\{\varphi_k\}$, we obtain an infinite system of differential equations

$$\dot{z}_k(t) + \lambda_k z_k(t) = -u_k(t) + v_k(t), \quad z_k(0) = z_{k0}, \quad k = 1, 2, \ldots$$

Integrating each equation of this system with the appropriate initial data, we obtain

$$z_k(t) = e^{-\lambda_k t}\left(z_{k0} - \int_0^t (u_k(s) - v_k(s)) e^{\lambda_k s} ds \right), \quad k = 1, 2, \ldots \tag{2.4}$$

where $z_{k0} = (z_0, \varphi_k)$ are the Fourier coefficients of the function z_0.

Define the pursuer's control as

$$u_i(t) = \begin{cases} u_{i0}(t) + v_i(t), & i = 1, 2, \ldots, \quad 0 \leqslant t \leqslant \vartheta \\ 0, & t > \vartheta \end{cases}, \quad u_{i0}(t) = e^{\lambda_i t} r_i(\vartheta) z_{i0}$$

That the pursuer's control we have constructed is admissible follows from the relations

$$F^{1/2}(\vartheta) \leqslant \left(\sum_{i=1}^{\infty} \lambda_i^r \int_0^{\vartheta} u_{i0}^2(t) dt \right)^{1/2} + G^{1/2}(\vartheta) \leqslant$$

$$\leqslant \left(\sum_{i=1}^{\infty} \lambda_i^r \int_0^{\vartheta} r_i^2(\vartheta) e^{2\lambda_i t} z_{i0}^2 dt \right)^{1/2} + \sigma = \Sigma^{1/2}(\vartheta) + \sigma = \rho - \sigma + \sigma = \rho$$

(we have used the definition of the number ϑ).

2. *The possibility completing the game.* Using (2.4), we have

$$z_k(\vartheta) = e^{-\lambda_k \vartheta}\left(z_{k0} - \int_0^{\vartheta} r_k(\vartheta) e^{\lambda_k s} e^{\lambda_k s} z_{k0} ds \right) = e^{-\lambda_k \vartheta}(z_{k0} - z_{k0}) = 0$$

3. *The construction of the evader's control.* We define the evader's control as follows:

$$v_i(t) = r_i(\vartheta) \frac{\sigma}{\rho - \sigma} e^{\lambda_i t} z_{i0}, \quad 0 \leqslant t \leqslant \vartheta, \quad \text{if} \quad \rho(t) > \sigma(t) \tag{2.5}$$

But if $\rho(\tau) = \sigma(\tau)$ for the first time at some time τ, $0 < \tau < \vartheta$, then the evader's control is identified with a control V_0 to be defined below (see formula (2.10)).

That the evader's control is admissible follows from the relations

$$G(t) \leq \frac{\sigma^2}{(\rho-\sigma^2)} \int_0^{\vartheta} \sum_{i=1}^{\infty} \lambda_i^r r_i^2(\vartheta) e^{2\lambda_i t} z_{i0}^2 dt = \frac{\sigma^2}{(\rho-\sigma)^2} \Sigma(\vartheta) = \sigma^2$$

and the construction of the control V_0.

The impossibility of completing the pursuit in the interval $[0,\vartheta)$

4. *Lemma*. Among all controls $u = u(t) = (u_1(t), u_2(t), \ldots)$, $0 \leq t \leq \tau$ such that

$$\int_0^{\tau} u_k(s) e^{\lambda_k s} ds = z_{k0}, \quad k = 1,2,\ldots \tag{2.6}$$

the control

$$u(t): \quad u_k(t) = r_k(\tau) e^{\lambda_k t} z_{k0}, \quad k = 1,2,\ldots \tag{2.7}$$

minimizes the functional $F(\tau)$.

Proof. Multiply both sides of equality (2.6) by $\lambda_k^r r_k(\tau) z_{k0}$ and sum over k. This gives

$$\Sigma(\tau) = \int_0^{\tau} \sum_{k=1}^{\infty} \lambda_k^r r_k(\tau) u_k(s) e^{\lambda_k s} z_{k0} ds \leq$$

$$\leq \left(\sum_{k=1}^{\infty} \lambda_k^r \int_0^{\tau} r_k^2(\tau) e^{2\lambda_k s} z_{k0}^2 ds \right)^{1/2} F^{1/2}(\tau) = \Sigma^{1/2}(\tau) F^{1/2}(\tau)$$

(where we have used the Cauchy–Bunyakovskii inequality). Consequently

$$\Sigma^{1/2}(\tau) \leq F^{1/2}(\tau) \tag{2.8}$$

Thus, inequality (2.8) is true for any control $u(t)$, $0 \leq t \leq \tau$ satisfying (2.6).

On the other hand, it can be verified that control (2.7) satisfies (2.8) with an equality sign.

Suppose the evader uses control (2.5) just constructed. We shall first show that, as long as the inequality $\rho(t) > \sigma(t)$, $t \in [0; \vartheta)$ is satisfied, the game cannot be completed. Suppose, on the contrary, that at some time t', $0 < t' < \vartheta$ for which

$$\rho(t') \geq \sigma(t') \tag{2.9}$$

the game is completed, that is, $z(t') = 0$.

Taking (2.5) into consideration, we infer from the equality $z(t') = 0$ that

$$\int_0^{t'} u_k(s) e^{\lambda_k s} ds = z_{k0} + \int_0^{t'} v_k(s) e^{\lambda_k s} ds =$$

$$= z_{k0} + \frac{\sigma}{\rho-\sigma} \int_0^{t'} r_k(\vartheta) e^{2\lambda_k s} z_{k0} ds = z_{k0} + \frac{\sigma}{\rho-\sigma} \frac{r_k(\vartheta)}{r_k(t')} z_{k0}$$

By the lemma, among all controls $u(t)$, $0 \leq t \leq t'$, satisfying this inequality, the control $u(t)$

$$u_k(t) = e^{\lambda_k t} \left(r_k(t') + \frac{\sigma}{\rho-\sigma} r_k(\vartheta) \right) z_{k0}, \quad 0 \leq t \leq t'$$

minimizes the functional $F(t')$. Therefore, taking the definition of the number ϑ into consideration, we have

$$F(t')-G(t')\geq \sum_{k=1}^{\infty} \lambda_k^r\left(r_k(t')+\frac{\sigma}{\rho-\sigma}r_k(\vartheta)\right)^2 z_{k0}^2\int_0^{t'} e^{2\lambda_k s}ds-$$

$$-\frac{\sigma^2}{(\rho-\sigma)^2}\sum_{k=1}^{\infty} \lambda_k^r r_k^2(\vartheta)\int_0^{t'} e^{2\lambda_k s}z_{k0}^2 ds=\sum_{k=1}^{\infty}\lambda_k^r\left(r_k(t')+2\frac{\sigma}{\rho-\sigma}r_k(\vartheta)\right)r_{k0}^2=$$

$$=\Sigma(t')+2\sigma(\rho-\sigma)>\Sigma(\vartheta)+2\sigma(\rho-\sigma)=\rho^2-\sigma^2$$

Consequently

$$\rho'-F(t')<\sigma^2-G(t')$$

or, what is the same, $\rho^2(t')<\sigma^2(t')$, contradicting inequality (2.9). Thus, the pursuer, while maintaining the inequality $\rho(t)>\sigma(t)$, cannot complete the game in the interval $[0,\vartheta)$.

Suppose $\rho(\tau)=\sigma(\tau)$ for the first time at some $\tau\in(0,\vartheta)$. Construct a control for the evader for $t\geq\tau$. Since $z(\tau)\neq 0$, it necessarily follows that for some components $z_k(\tau)\neq 0$, $k\in\{1,2,\ldots\}$. We may assume, without loss of generality, that $z_1(\tau)\neq 0$. To fix our ideas, suppose that $z_1(\tau)>0$. Put

$$t_i=\tau+ih,\ i=0,1,\ldots,n,\ t_0=\tau,\ t_n=\vartheta;\ \alpha_i=\left(\int_{t_i}^{t_{i+1}}|u_1(s)|^2\,ds\right)^{1/2},\ i=0,1,\ldots,n-1$$

where h is a positive number to be chosen later. The evader's control for $t\geq\tau$ is now defined as

$$v_1(t)=\begin{cases}0, & t_0\leq t\leq t_1,\\ (\alpha_{i-1}e^{\lambda_1 t})/g_i, & t_i<t\leq t_{i+1},\ i=1,2,\ldots\\ 0, & t\geq\vartheta\end{cases}\tag{2.10}$$

$$v_k(t)=0,\ k=2,3,\ldots$$

where

$$g_i=\left(\int_{t_i}^{t_{i+1}}e^{2\lambda_1 s}ds\right)^{1/2}=\left(\frac{e^{2\lambda_1 h}-1}{2\lambda_1}\right)^{1/2}e^{\lambda_1 t_i},\ i=0,1,2,\ldots,n\tag{2.11}$$

Denote this evader's control by V_0. Then for $t\in[t_0,t_1)$, using the Cauchy–Bunyakovskii inequality, we have

$$z_1(t)=z_1(\tau)-\int_{t_0}^{t}u_1(s)e^{\lambda_1 s}ds\geq z_1(\tau)-\left(\int_{t_0}^{t}u_1^2(s)ds\int_{t_0}^{t}e^{2\lambda_1 s}ds\right)^{1/2}\geq z_1(\tau)-g_0\rho\tag{2.12}$$

For $t_1<t<\vartheta$, by (2.10), we have (assuming that $t\in[t_k;t_{k+1}]\subset[t;\vartheta]$)

$$z_1(t)=z_1(\tau)+\int_{t_0}^{t}[v_1(s)-u_1(s)]e^{\lambda_1 s}ds=$$

$$=z_1(\tau)+\sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}}v_1(s)e^{\lambda_1 s}ds-\sum_{i=0}^{k-1}\int_{t_i}^{t_{i+1}}u_1(s)e^{\lambda_1 s}ds+\int_{t_k}^{t}v_1(s)e^{\lambda_1 s}ds-$$

$$-\int_{t_k}^{t}u_1(s)e^{\lambda_1 s}ds\geq z_1(\tau)+\sum_{i=1}^{k-1}g_i\alpha_{i-1}-\sum_{i=0}^{k-1}\left(\int_{t_i}^{t_{i+1}}u_1^2(s)ds\int_{t_i}^{t_{i+1}}e^{2\lambda_1 s}ds\right)^{1/2}-$$

$$-\left(\int_{t_k}^{t}u_1^2(s)ds\int_{t_k}^{t}e^{2\lambda_1 s}ds\right)^{1/2}\geq z_1(\tau)+\sum_{i=1}^{k}g_i\alpha_{i-1}-\sum_{i=1}^{k}g_{i-1}\alpha_{i-1}-g_k(\alpha_k+\alpha_{k-1})\geq$$

$$\geq z_1(\tau)-\sum_{i=1}^{k}\Delta_{i-1}\alpha_{i-1}-2g_k\rho;\ \Delta_i=g_i-g_{i+1}\tag{2.13}$$

Using relations (2.11), we have

$$|\Delta_{i-1}| = \left(\frac{e^{2\lambda_1 h} - 1}{2\lambda_1}\right)^{1/2} (e^{\lambda_1 h} - 1)e^{\lambda_1 t_{i-1}} \leqslant Ch^{3/2} e^{\lambda_1 \vartheta} \tag{2.14}$$

where C is some positive number. Then, bearing in mind the inequalities

$$kh \leqslant \vartheta,\ \alpha_i \leqslant \rho$$

we infer from relations (2.13) and (2.14) that

$$z_1(t) \geqslant z_1(\tau) - Ch^{3/2} \sum_{i=1}^{k} e^{\lambda_1 \vartheta} \alpha_{i-1} - 2g_k\rho \geqslant z_1(\tau) - (Ce^{\lambda_1 \vartheta} \vartheta h^{1/2} + 2g_k)\rho \tag{2.15}$$

The constant h may be chosen so that expressions (2.12) and (2.15) exceed $z_1(\tau)/2$. Then for such h

$$z_1(t) \geqslant z_1(\tau)/2 > 0,\ t \in [\tau, \vartheta]$$

In addition,

$$\sum_{i=1}^{\infty} J_i(\tau, t) = J_1(\tau, t) \leqslant \lambda_1^r \sum_{i=1}^{k} \int_{t_i}^{t_{i+1}} v_1^2(s)ds \leqslant \lambda_1^r \sum_{i=1}^{k} \alpha_{i-1}^2 \leqslant \sum_{i=1}^{k} I_1(t_{i-1}, t_i) =$$

$$= I_1(\tau, t_k) \leqslant \sum_{k=1}^{\infty} I_k(\tau, t) \leqslant \rho^2(\tau) = \sigma^2(\tau)$$

$$I_k(\tau, t) = \lambda_k^r \int_{\tau}^{t} |u_k(s)|^2\, ds, \quad J_k(\tau, t) = \lambda_k^r \int_{\tau}^{t} |v_k(s)|^2\, ds$$

Consequently, control (2.10) is admissible.

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